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Inner product and Gegenbauer polynomials in Sobolev space

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Abstract. In this paper we consider the system of functions $G_{r,n}^{\alpha}(x)$ $(\ r\in\mathbb{N},\ \ n=0,1,\dots)$ which is orthogonal with respect to the Sobolev-type inner product on $(-1,1)$ and generated by orthogonal Gegenbauer polynomials. The main goal of this work is to study some properties related to the system $\{\varphi_{k,r}(x)\}_{k\geq 0}$ of the functions generated by the orthogonal system $\{G_{r,n}^{\alpha}(x)\}$ of Gegenbauer functions. We study the conditions on a function $f(x)$ given in a generalized Gegenbauer orthogonal system for it to be expandable into a generalized mixed Fourier series of the form

$$f(x)\sim\sum_{k=0}^{r-1}f^{(k)}(-1)\frac{(x+1)^k}{k!}+\sum_{k=r}^{\infty}C_{r,k}^{\alpha}(f)\varphi_{r,k}^{\alpha}(x),$$

as well as the convergence of this Fourier series. The second result of this paper is the proof of a recurrence formula for the system $\{\varphi_{k,r}(x)\}_{k\geq 0}$. We also discuss the asymptotic properties of these functions, and this represents the latter result of our contribution.

Keywords: inner product, Sobolev space, Gegenbauer polynomials

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Скалярное произведение и многочлены Гегенбауэра в пространстве Соболева

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Аннотация. В данной работе рассматривается система функций $G^{\alpha}_{r,n}(x)$ ($r \in \mathbb{N}$, $n = 0, 1, \dots$), которые ортогональны относительно скалярного произведения соболевского типа на $(-1,1)$ и порождены ортогональными полиномами Гегенбауэра. Основной целью данной работы является изучение некоторых свойств, связанных с системой $\{\varphi_{k,r}(x)\}_{k\geq 0}$ функций, порожденных ортогональной системой $\{G^{\alpha}_{r,n}(x)\}$ функций Гегенбауэра. Исследуются условия на функцию $f(x)$, заданную в обобщенной ортогональной системе Гегенбауэра, которые гарантируют ее разложимость в обобщенный смешанный ряд Фурье вида

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(-1)\ frac(x+1)^k k! + \sum_{k=r}^{\infty} C^{\alpha}_{r,k}(f)\varphi^{\alpha}_{r,k}(x),$$

и изучается сходимость этого ряда Фурье. Второй результат этой статьи состоит в доказательстве рекуррентной формулы для системы $\{\varphi_{k,r}(x)\}_{k\geq 0}$. Мы также обсуждаем асимптотические свойства этих функций, что составляет заключительный результат нашей работы.

Ключевые слова: скалярное произведение, пространство Соболева, многочлены Гегенбауэра

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Introduction

Consider an orthogonal system $\{\varphi_k(x)\}_{k\geq 0}$ on (a,b) with $\rho(x)$ as a weight function, and let $r\in\mathbb{N}$. We construct a new orthogonal system $\{\varphi_{k,r}(x)\}_{k\geq 0}$ following the Sobolev-type inner product:

$$\langle f,g\rangle_S=\sum_{v=0}^{r-1} f^{(v)}(a)g^{(v)}(b)+\int_a^b f^{(r)}(t)g^{(r)}(t)\rho(t)dt. \tag{0.1}$$

Quite a few authors have presented this type of construction, see, for example, the works of R. M. Gadzhimirzaev ([1]) and I. I. Sharapudinov ([2–6]) on the construction of mixed Fourier series. The author in his works presents some particular cases of systems generated by classes of orthogonal functions, namely Jacobi, Legendre, Chebychev, Laguerre, and Haar.

Gegenbauer polynomials are widely used in several fields, they are of a particular interest in applications. It is clear that these polynomials present a special case of those of Jacobi for particular values of parameters. In this work, we will reconstruct the orthogonal system $\{\varphi_{k,r}(x)\}_{k\geq 0}$ generated by the Gegenbauer polynomials using the approach which is different from the one used by I. Sharapudinov.

Denote by $L^p_\rho(a,b)$ the space of measurable functions $f(x)$, $x\in(a,b)$, with

$$\int_a^b f^{(r)}(t)g^{(r)}(t)\rho(t)dt<\infty.$$

When $\rho(x)=1$, we write $L^p_\rho(a,b)=L^p(a,b)$. It is clear that $L^p_\rho(a,b)$ is the Banach space with the norm

$$\|f\|_{p,\rho}=\left(\int_a^b |f(x)|^p\,\rho(x)dx\right)^{\frac{1}{p}}.$$

We can define the functions of the system $\{\varphi_{k,r}(x)\}_{k\geq 0}$ as follows [5]:

$$\begin{cases} \varphi_{r,r+k}(x)=\dfrac{1}{(r-1)!}\displaystyle\int_a^b (x-t)^{r-1}\varphi_k(t)dt, & k=0,1,2,...\\ \varphi_{r,k}(x)=\dfrac{(x-a)^k}{k!}, & k=0,1,2,\cdots,r-1. \end{cases} \tag{0.2}$$

From (0.2) for $x\in(a,b)$, we have

$$\varphi^{(v)}_{r,k}(x)=\begin{cases} \varphi_{r-v,k-v}(x), & \text{if } 0\leq v\leq r-1,\ r\leq k,\\ \varphi_{k-v}(x), & \text{if } v=r\leq k,\\ \varphi_{r-v,k-v}(x), & \text{if } v\leq k<r,\\ 0, & \text{if } k<v\leq r. \end{cases}$$

Denote by $W^r_{L^p_\rho(a,b)}$ the Sobolev weighted space. This space consists of all $r-1$ times continuously differentiable on the interval $[a,b]$ functions $f(x)$ such that $f^{(r-1)}(x)$ is absolutely continuous on $[a,b]$ and $f^{(r)}(x)\in L^p_\rho(a,b)$.

For $p=2$, we define in $W^r_{L^2_\rho(a,b)}$ the inner product by (0.1). We can set the norm for any function $f\in W^r_{L^2_\rho(a,b)}$ by

$$\|f\|_{W^r_{L^2_\rho(a,b)}}=\sqrt{\langle f,f\rangle_S},$$

which allows us to deduce that $\left(W^r_{L^2_\rho(a,b)}, \|\cdot\|_{W^r_{L^2_\rho(a,b)}}\right)$ is the Banach space, and $\left(W^r_{L^2_\rho(a,b)}, \langle\cdot,\cdot\rangle_S\right)$ is the Hilbert space.

The system $\{\varphi_{r,k}(x)\}_{k\geq 0}$ is said to be Sobolev-orthogonal according to the inner product (0.1) generated by the orthonormal system $\{\varphi_k(x)\}_{k\geq 0}$.

1. Main concepts: some properties of Gegenbauer polynomials

Let $\varphi^\alpha_{r,n}(x)$ be the Sobolev-orthogonal polynomials according to the inner product

$$\langle f,g\rangle = \sum_{v=0}^{r-1} f^{(v)}(-1)g^{(v)}(-1) + \int_{-1}^{1} f^{(r)}(t)g^{(r)}(t)w(t)dt,$$

where $w(t) = (1-t^2)^{\alpha-\frac{1}{2}}$.

Here are some properties of the Gegenbauer polynomials:

- Gegenbauer polynomials are given by [7]

$$g^\alpha_n(x) = \frac{(2\alpha)_n}{\left(\alpha+\frac{1}{2}\right)_n} P_n^{\left(\alpha-\frac{1}{2},\alpha-\frac{1}{2}\right)}(x),$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial, $(\alpha)_n = \alpha\,(\alpha+1)\,(\alpha+2)\ldots(\alpha+n-1)$. We can have the following formula

$$g^\alpha_n(x) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(2\alpha+n\right)}{\Gamma\left(2\alpha\right)\Gamma\left(\alpha+n+\frac{1}{2}\right)} P_n^{\left(\alpha-\frac{1}{2},\alpha-\frac{1}{2}\right)}(x),$$

with

$$g^\alpha_0(x) = 1, \quad g^\alpha_1(x) = 2\alpha x.$$

- Recurrence formula:

$$g^\alpha_n(x) = \frac{1}{n}\left(2x\left(\alpha+n-1\right)g^\alpha_{n-1}(x) - \left(n+2\alpha-2\right)g^\alpha_{n-2}(x)\right).$$

- Orthogonality formula:

$$\int_{-1}^{1} g^\alpha_n(x)g^\alpha_m(x)(1-x^2)^{\alpha-\frac{1}{2}}dx = \delta_{nm}h^\alpha_n,$$

where

$$h^\alpha_n = \pi\frac{2^{1-2\alpha}\Gamma\left(n+2\alpha\right)}{n!\left(n+\alpha\right)\Gamma^2\left(\alpha\right)}.$$

Let us put

$$G^\alpha_n(x) = \sqrt{\frac{w(x)}{h^\alpha_n}}\,g^\alpha_n(x),$$

where

$$w(x) = (1-x^2)^{\alpha-\frac{1}{2}}.$$

$G^\alpha_n(x)$ are the Gegenbauer functions [7, p. 776]. The system $\{G^\alpha_n(x)\}$ is orthogonal on $(-1,1)$, i. e.

$$\int_{-1}^{1} G^\alpha_n(x)G^\alpha_m(x)dx = \delta_{nm}.$$

- Some values:

$$G_0^\alpha(x) = \sqrt{\frac{w(x)\alpha\Gamma^2(\alpha)}{\pi 2^{1-2\alpha}\Gamma(\alpha)}}, \qquad G_1^\alpha(x) = 2\alpha x\sqrt{\frac{w(x)(1+\alpha)\Gamma^2(\alpha)}{\pi 2^{1-2\alpha}\Gamma(1+\alpha)}}.$$

1.1. Orthogonal Sobolev functions generated by the Gegenbauer functions $G_n^\alpha(x)$

D e f i n i t i o n 1.1. For $r \in \mathbb{N}$, we define the functions $\varphi_{r,k}^\alpha(x)$ $(k = 0, 1, \dots)$ by

$$\begin{cases} \varphi_{r,k}^\alpha(x) = \dfrac{(x+1)^k}{k!}, \quad k = 0, 1, ...r-1, \\ \varphi_{r,k+r}^\alpha(x) = \dfrac{1}{(r-1)!}\displaystyle\int_{-1}^{x}(x-t)^{r-1}G_k^\alpha(t)dt, \quad k = 0, 1, \dots. \end{cases}$$

We will calculate the functions $\varphi_{r,k+r}^\alpha(x)$ for any $k \in \mathbb{N}$ and $x \in [-1, 1]$.

Theorem 1.1 (Fisrt aim result)**.** *For $\alpha > -1$, we have the following relations:*

1. $\varphi_{r,r+n}^\alpha(x)$

$$= -2r\sqrt{1+\frac{\alpha}{n}}\varphi_{r+1,r+n}^\alpha(x) + 2x\sqrt{1+\frac{\alpha}{n}}\varphi_{r,r+n-1}^\alpha(x) - \Big(1 - \frac{1}{n+\alpha}\Big)\sqrt{1+\frac{\alpha}{n}}\varphi_{r,r+n-2}^\alpha(x).$$

2. $$\varphi_{r,r}^\alpha(x) = -2r\sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(2\alpha)}}\varphi_{r,r+2}^\alpha(x) + 2x\sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(2\alpha)}}\varphi_{r,r+1}^\alpha(x) - \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(2\alpha)}}\varphi_{r,r+2}^\alpha(x).$$

3. $$\varphi_{1,1+n}^\alpha(x) = -2\sqrt{1+\frac{\alpha}{n}}\varphi_{2,1+n}^\alpha(x) + 2x\sqrt{1+\frac{\alpha}{n}}\varphi_{1,n}^\alpha(x) - \Big(1 - \frac{1}{n+\alpha}\Big)\sqrt{1+\frac{\alpha}{n}}\varphi_{1,n-1}^\alpha(x).$$

To prove this theorem, we have the following lemma:

Lemma 1.1. [8, p. 80–83] *Here are the formulas for the derivations of the Gegenbauer functions:*

1. $$\frac{d}{dx}g_n^\alpha(x) = 2\alpha g_{n-1}^\alpha(x).$$

2. $$(1-x^2)\frac{d}{dx}g_n^\alpha(x) = \frac{2}{n+\alpha}\left\{(n+2\alpha-1)(n+2\alpha)g_{n-1}^\alpha(x) - n(n+1)g_{n+1}^\alpha(x)\right\}$$
$$= -nxg_n^\alpha(x) + (n+\alpha-1)g_{n-1}^\alpha(x)$$
$$= (n+2\alpha)xg_n^\alpha(x) - (n+1)g_{n+1}^\alpha(x).$$

3. $$ng_n^\alpha(x) = x\frac{d}{dx}g_n^\alpha(x) - \frac{d}{dx}g_{n-1}^\alpha(x).$$

4. $$\frac{d}{dx}\left[g_{n+1}^\alpha(x) - g_{n-1}^\alpha(x)\right] = 2(n+\alpha)g_n^\alpha(x) = 2\alpha\left[g_n^{\alpha+1}(x) - g_{n-2}^{\alpha+1}(x)\right].$$

P r o o f. (of theorem 1.1): *1.* Firstly, its clear that:

$$\varphi^{\alpha}_{0,n}(x) = G^{\alpha}_{n}(x), \quad \varphi^{\alpha}_{1,0}(x) = 1, \quad \varphi^{\alpha}_{1,1}(x) = \int_{-1}^{x} G^{\alpha}_{0}(t)dt.$$

We have

$$\varphi^{\alpha}_{r,r+n}(x) = \frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}G^{\alpha}_{n}(t)dt, \tag{1.1}$$

where

$$G^{\alpha}_{n}(t) = \frac{\sqrt{w(t)}}{\sqrt{h^{\alpha}_{n}}}g^{\alpha}_{n}(t).$$

By Lemma 1.1,

$$g^{\alpha}_{n}(x) = \frac{2x(\alpha+n-1)}{n}g^{\alpha}_{n-1}(x) - \frac{n+2\alpha-2}{n}g^{\alpha}_{n-2}(x). \tag{1.2}$$

Then (1.1) becomes

$$\begin{aligned}
\varphi^{\alpha}_{r,r+n}(x) &= \frac{2(\alpha+n-1)}{(r-1)!n\sqrt{h^{\alpha}_{n}}}\int_{-1}^{x}\sqrt{w(t)}t\,(x-t)^{r-1}\,g^{\alpha}_{n-1}(t)dt \\
&\quad -\frac{\alpha+n-1}{(r-1)!n\sqrt{h^{\alpha}_{n}}}\int_{-1}^{x}\sqrt{w(t)}\,(x-t)^{r-1}\,g^{\alpha}_{n-2}(t)dt \\
&= \frac{2(\alpha+n-1)\sqrt{h^{\alpha}_{n-1}}}{(r-1)!n\sqrt{h^{\alpha}_{n}}}\int_{-1}^{x}\frac{\sqrt{w(t)}}{\sqrt{h^{\alpha}_{n-1}}}t\,(x-t)^{r-1}\,g^{\alpha}_{n-1}(t)dt \\
&\quad -\frac{(\alpha+n-1)\sqrt{h^{\alpha}_{n-2}}}{(r-1)!n\sqrt{h^{\alpha}_{n}}}\int_{-1}^{x}\frac{\sqrt{w(t)}}{\sqrt{h^{\alpha}_{n-2}}}\,(x-t)^{r-1}\,g^{\alpha}_{n-2}(t)dt \\
&= \frac{2(\alpha+n-1)\sqrt{h^{\alpha}_{n-1}}}{(r-1)!n\sqrt{h^{\alpha}_{n}}}\int_{-1}^{x}tG^{\alpha}_{n-1}(t)\,(x-t)^{r-1}\,dt \\
&\quad -\frac{(\alpha+n-1)\sqrt{h^{\alpha}_{n-2}}}{(r-1)!n\sqrt{h^{\alpha}_{n}}}\int_{-1}^{x}G^{\alpha}_{n-2}(t)\,(x-t)^{r-1}\,dt.
\end{aligned}$$

After simplification of the terms without integral signs, we obtain

$$\begin{aligned}
\varphi^{\alpha}_{r,r+n}(x) &= 2\sqrt{1+\frac{\alpha}{n}}\frac{1}{(r-1)!}\int_{-1}^{x}t(x-t)^{r-1}G^{\alpha}_{n-1}(t)dt \\
&\quad -\left(1-\frac{1}{n+\alpha-1}\right)\sqrt{1+\frac{\alpha}{n}}\frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}G^{\alpha}_{n-2}(t)dt.
\end{aligned}$$

Put

$$\varphi^{\alpha}_{r,r+n-2}(x) = \frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}G^{\alpha}_{n-2}(t)dt.$$

Still to be calculated

$$J = \frac{1}{(r-1)!}\int_{-1}^{x}t(x-t)^{r-1}G^{\alpha}_{n-1}(t)dt.$$

We have

$$\begin{aligned} J &= \frac{1}{(r-1)!}\int_{-1}^{x}(t-x+x)(x-t)^{r-1}G_{n-1}^{\alpha}(t)dt \\ &= -\frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r}G_{n-1}^{\alpha}(t)dt+\frac{x}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}G_{n-1}^{\alpha}(t)dt \\ &= -r\varphi_{r+1,r+n}^{\alpha}(x)+x\varphi_{r,r+n-1}^{\alpha}(x). \end{aligned}$$

Thus

$$\begin{aligned} &\varphi_{r,r+n}^{\alpha}(x) \\ &\quad= -2r\sqrt{1+\frac{\alpha}{n}}\varphi_{r+1,r+n}^{\alpha}(x)+2x\sqrt{1+\frac{\alpha}{n}}\varphi_{r,r+n-1}^{\alpha}(x)-\Big(1-\frac{1}{n+\alpha-1}\Big)\sqrt{1+\frac{\alpha}{n}}\varphi_{r,r+n-2}^{\alpha}(x). \end{aligned}$$

2. We use the following relation:

$$\varphi_{r,r}^{\alpha}(x)=\frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}G_{0}^{\alpha}(t)dt=\frac{1}{(r-1)!}\frac{1}{\sqrt{h_0^{\alpha}}}\int_{-1}^{x}\sqrt{w(t)}g_0^{\alpha}(t)(x-t)^{r-1}dt.$$

By lemma 1.1 (formula 4), we have

$$g_0^{\alpha}(t)=\frac{2(\alpha+1)x}{\alpha}g_1^{\alpha}(t)-\frac{2}{\alpha}g_2^{\alpha}(t),$$

so

$$\begin{aligned} \varphi_{r,r}^{\alpha}(x) &= \frac{2(\alpha+1)}{\alpha(r-1)!}\int_{-1}^{x}\sqrt{\frac{w(t)}{h_0^{\alpha}}}tg_1^{\alpha}(t)(x-t)^{r-1}dt-\frac{2}{\alpha(r-1)!}\int_{-1}^{x}\sqrt{\frac{w(t)}{h_0^{\alpha}}}g_2^{\alpha}(t)(x-t)^{r-1}dt \\ &= \frac{2(\alpha+1)}{\alpha(r-1)!}\sqrt{\frac{h_1^{\alpha}}{h_0^{\alpha}}}\int_{-1}^{x}t(x-t)^{r-1}G_1^{\alpha}(t)dt-\frac{2}{\alpha(r-1)!}\sqrt{\frac{h_2^{\alpha}}{h_0^{\alpha}}}\int_{-1}^{x}(x-t)^{r-1}G_2^{\alpha}(t)dt. \end{aligned}$$

Put

$$\varphi_{r,r+2}^{\alpha}(x)=\frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}G_2^{\alpha}(t)dt.$$

Let us calculate

$$\begin{aligned} H &= \frac{1}{(r-1)!}\int_{-1}^{x}t(x-t)^{r-1}G_1^{\alpha}(t)dt \\ &= \frac{1}{(r-1)!}\int_{-1}^{x}(t-x+x)\,(x-t)^{r-1}\,G_1^{\alpha}(t)dt \\ &= \frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r}G_1^{\alpha}(t)dt+\frac{x}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}G_1^{\alpha}(t)dt \\ &= -r\varphi_{r+1,r+2}^{\alpha}(x)+x\varphi_{r,r+1}^{\alpha}(x). \end{aligned}$$

Then

$$\varphi_{r,r}^{\alpha}(x)=\frac{2(\alpha+1)}{\alpha(r-1)!}\sqrt{\frac{h_1^{\alpha}}{h_0^{\alpha}}}\left(-r\varphi_{r+1,r+2}^{\alpha}(x)+x\varphi_{r,r+1}^{\alpha}(x)\right)-\frac{2}{\alpha}\sqrt{\frac{h_2^{\alpha}}{h_0^{\alpha}}}\varphi_{r,r+2}^{\alpha}(x),$$

hence

$$\varphi_{r,r}^{\alpha}(x)=2\sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(2\alpha)}}\left(-r\varphi_{r+1,r+2}^{\alpha}(x)+x\varphi_{r,r+1}^{\alpha}(x)\right)-\sqrt{\frac{\Gamma(2+2\alpha)}{\Gamma(2\alpha)}}\varphi_{r,r+2}^{\alpha}(x).$$

So we obtain the second formula.

3. It is sufficient to replace $r=1$ in the first formula, since it represents a special case. □

1.2. Problem of mixed Fourier series

Let $f \in W^r_{L^2_\rho(-1,1)}$. If this function is given in the generalized Gegenbauer orthogonal system $\{\varphi^\alpha_{r,n}(x)\}_{n=0}^\infty$, then

$$f(x) \sim \sum_{k=0}^{\infty} C^\alpha_{r,k}(f)\varphi^\alpha_{r,k}(x). \tag{1.3}$$

This Fourier series will have the form

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(-1)\frac{(x+1)^k}{k!} + \sum_{k=r}^{\infty} C^\alpha_{r,k}(f)\varphi^\alpha_{r,k}(x), \tag{1.4}$$

with

$$\begin{aligned} C^\alpha_{r,k}(f) = \hat{f}_{r,k} &= \int_{-1}^{1} f^{(r)}(t)\varphi^\alpha_{r,k}(t)dt \\ &= \int_{-1}^{1} f^{(r)}(t)G^\alpha_{k-r}(t)dt, \quad k = r, r+1, ..., \end{aligned}$$

called the Fourier coefficient. For $r = 0$, we have

$$f(x) \sim \sum_{k=r}^{\infty} C^\alpha_{0,k}(f)\varphi^\alpha_{0,k}(x) \sim \sum_{k=r}^{\infty} \hat{f}_{0,k}\varphi^\alpha_{0,k}(x),$$

with

$$\hat{f}_{0,k} = \int_{-1}^{1} f(t)G^\alpha_k(t)dt. \tag{1.5}$$

In this section, we will give the expressions of (1.4) and (1.5) taking into account the expression of $G^\alpha_n(t)$. Also we will prove the convergence of the series (1.3). The following result is similar to the one given in [5].

Theorem 1.2. *For $\alpha > -1$, the system of functions $\{\varphi^\alpha_{r,n}(x)\}$ generated by the Gegenbauer functions $G^\alpha_n(x)$ and given by the formula*

$$\varphi^\alpha_{r,n+r}(x) = \frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}G^\alpha_n(t)dt, \quad n \geq 0,$$

is complete in $W^r_{L^2_\rho(-1,1)}$ and orthonormal via Sobolev's inner product

$$\langle f, g\rangle = \sum_{v=0}^{r-1} f^{(v)}(-1)\, g^{(v)}(-1) + \int_{-1}^{1} f^{(r)}(t)g^{(r)}(t)w(t)dt.$$

It follows from the two formulas

$$\begin{cases} \varphi^\alpha_{r,r+n}(x) = \dfrac{1}{(r-1)!}\displaystyle\int_{-1}^{x}(x-t)^{r-1}G^\alpha_n(t)dt, & n \geq 0, \\ \varphi^\alpha_{r,n}(x) = \dfrac{(x+1)^n}{n!}, & n = 0, 1, \ldots r-1, \end{cases}$$

that for all $x \in (-1, 1)$,

$$\left(\varphi^\alpha_{r,n}(x)\right)^{(v)} = \begin{cases} \varphi^\alpha_{r-v,n-v}(x), & if \;\; 0 \leq v \leq r-1, \quad r \leq n, \\ G^\alpha_{n-v}(x), & if \;\; v = r \leq n, \\ \varphi^\alpha_{r-v,n-v}(x), & if \;\; v \leq n < r, \\ 0, & if \;\; n < v \leq r, \end{cases}$$

with $\varphi^\alpha_{0,n}(x) = G^\alpha_n(x)$.

1.2.1. Study of the convergence of the series (1.4)

Let $f \in W^r_{L^2_\rho(-1,1)}$, then $f^{(r)} \in L^p$ with

$$f^{(r)}(x) \sim \sum_{k=0}^{\infty} \hat{C}_{r,k}\left(f^{(r)}\right) G_k^\alpha(x),$$

where

$$\hat{C}_{r,k}\left(f^{(r)}\right) = \int_{-1}^{1} f^{(r)}(t) G_k^\alpha(t) dt, \quad \text{for all } k \geq 0.$$

We will prove the following result:

Theorem 1.3 (Second aim result). *For $\alpha > 0$, $x \in (-1, A]$, $(A < 1)$, and $f \in W^r_{L^p}$ with $\frac{4}{3} < p < 4$, the Fourier series*

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(-1)\frac{(x+1)^k}{k!} + \sum_{k=r}^{\infty} C^\alpha_{r,k}(f)\varphi^\alpha_{r,k}(x)$$

converges uniformly to the function f.

P r o o f. We note the following partial sums:

$$S^\alpha_{r,n}(f,x) = \sum_{k=0}^{r-1} f^{(k)}(-1)\frac{(x+1)^k}{k!} + \sum_{k=r}^{n} C^\alpha_{r,k}(f)\varphi^\alpha_{r,k}(x),$$

$$S^\alpha_{n}\left(f^{(r)},x\right) = \sum_{k=0}^{n} \hat{C}_{r,k}\left(f^{(r)}\right) G_k^\alpha(x).$$

Then

$$\left|f(x) - S^\alpha_{r,n+r}(f,x)\right| = \left|\frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1} f^{(r)}(t)dt - \sum_{k=r}^{r+n} C^\alpha_{r,k}(f)\varphi^\alpha_{r,k}(x)\right|$$

with

$$\varphi^\alpha_{r,k+r}(x) = \frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1} G_n^\alpha(t)dt,$$

so

$$\varphi^\alpha_{r,k}(x) = \frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1} G_{k-r}^\alpha(t)dt.$$

Hence,

$$\begin{aligned}
|f(x) \ - \ & S^\alpha_{r,n+r}(f,x)| \\
&= \left|\frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1} f^{(r)}(t)dt - \frac{1}{(r-1)!}\sum_{k=r}^{r+n} C^\alpha_{r,k}(f)\int_{-1}^{x}(x-t)^{r-1} G_{k-r}^\alpha(t)dt\right| \\
&= \frac{1}{(r-1)!}\left|\int_{-1}^{x}(x-t)^{r-1}\left(f^{(r)} - \sum_{k=r}^{r+n} C^\alpha_{r,k}(f) G_{k-r}^\alpha(t)\right)dt\right|
\end{aligned}$$

with

$$\sum_{k=r}^{r+n} C_{r,k}^{\alpha}(f) G_{k-r}^{\alpha}(t) = S_n^{\alpha}\left(f^{(r)}, x\right).$$

Then

$$\begin{aligned} \left|f(x) - S_{r,n+r}^{\alpha}(f,x)\right| &= \frac{1}{(r-1)!}\left|\int_{-1}^{x}(x-t)^{r-1}\left(f^{(r)}(t) - S_n^{\alpha}\left(f^{(r)},t\right)\right)dt\right| \\ &\le \frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}\left|f^{(r)}(t) - S_n^{\alpha}\left(f^{(r)},t\right)\right|dt. \end{aligned}$$

Using Hölder's inequality, we get:

$$\left|f(x) - S_{r,n+r}^{\alpha}(f,x)\right| \le \frac{1}{(r-1)!}\left(\int_{-1}^{x}(x-t)^{(r-1)q}dt\right)^{\frac{1}{q}}\left(\int_{-1}^{x}\left|f^{(r)}(t) - S_n^{\alpha}\left(f^{(r)},t\right)\right|^{p}dt\right)^{\frac{1}{p}}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Calculate

$$J = \int_{-1}^{x}(x-t)^{(r-1)q}dt = (-1)^{q(r-1)}\int_{-1}^{x}(x-t)^{q(r-1)}dt = \frac{(1+x)^{q(r-1)+2}}{q(r-1)+1} \quad \text{for } x \in (-1, A].$$

Then

$$\left|f(x) - S_{r,n+r}^{\alpha}(f,x)\right| \le \frac{1}{(r-1)!}\left(\frac{(1+A)^{q(r-1)+2}}{q(r-1)+1}\right)^{\frac{1}{q}}\left\|f^{(r)}(x) - S_n^{\alpha}\left(f^{(r)},x\right)\right\|_{L^p}.$$

Since

$$\left\|f^{(r)}(x) - S_n^{\alpha}\left(f^{(r)},x\right)\right\|_{L^p} \to 0 \text{ as } n \to \infty,$$

it results that

$$\left|f(x) - S_{r,n+r}^{\alpha}(f,x)\right| \to 0$$

uniformly on $(-1, A]$. □

Theorem 1.4. *Suppose that* $-\frac{1}{2} < \alpha < \frac{3}{2}$. *If* $f \in W^{r}_{L^2_{\rho}(-1,1)}$, *then we have the uniform convergence of the Fourier series*

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(-1)\frac{(x+1)^k}{k!} + \sum_{k=r}^{\infty} C_{r,k}^{\alpha}(f)\varphi_{r,k}^{\alpha}(x)$$

on $(-1,1)$ *to the function* f.

2. Asymptotic forms of the functions $\varphi_{1,1+n}^{\alpha}(x)$

We say that

$$\varphi_{1,1+n}^{\alpha}(x) = \int_{-1}^{x} G_n^{\alpha}(t)dt = \frac{1}{\sqrt{h_n^{\alpha}}}\int_{-1}^{x}\sqrt{w(t)}g_n^{\alpha}(t)dt,$$

where

$$w(t) = (1-t^2)^{\alpha-\frac{1}{2}}, \qquad h_n^{\alpha} = \pi\frac{2^{1-2\alpha}\Gamma(1+2\alpha)}{n!(n+\alpha)\left[\Gamma(\alpha)\right]^2}.$$

Then

$$\varphi_{1,1+n}^{\alpha}(x)=\frac{1}{\sqrt{h_n^{\alpha}}}\int_{-1}^{x}(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}}g_n^{\alpha}(t)dt.$$

Integrating by parts and using the first formula of Lemma 1.1, we get:

$$\varphi_{1,1+n}^{\alpha}(x)=\left|\begin{array}{ll} u=(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}} & du=\left(\frac{\alpha}{2}-\frac{1}{4}\right)(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}-1}(-2t)dt \\ dv=g_n^{\alpha}(t)dt & v=\frac{1}{2(\alpha-1)}g_{n+1}^{\alpha-1}(t)\end{array}\right|,$$

so

$$\varphi_{1,1+n}^{\alpha}(x)=\frac{1}{2\left(\alpha-1\right)\sqrt{h_n^{\alpha}}}(1-x^2)^{\frac{\alpha}{2}-\frac{1}{4}}g_{n+1}^{\alpha-1}(x)+\frac{\frac{\alpha}{2}-\frac{1}{4}}{\left(\alpha-1\right)\sqrt{h_n^{\alpha}}}\int_{-1}^{x}t(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}-1}g_{n+1}^{\alpha-1}(t)dt,$$

$$\varphi_{1,1+n}^{\alpha}(x)=\left|\begin{array}{ll} u=(1-t^2)^{\frac{\alpha}{2}-\frac{9}{4}} & du=-2t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}}\left(\frac{1}{2}\alpha-\frac{5}{4}\right)dt \\ dv=g_{n+1}^{\alpha-1}(t)dt & v=\frac{1}{2(\alpha-2)}g_{n+2}^{\alpha-2}(t)\end{array}\right|.$$

Then

$$\varphi_{1,1+n}^{\alpha}(x)=\frac{1}{2\left(\alpha-1\right)\sqrt{h_n^{\alpha}}}(1-x^2)^{\frac{\alpha}{2}-\frac{1}{4}}g_{n+1}^{\alpha-1}(x)+\frac{(1-x^2)^{\frac{\alpha}{2}-\frac{9}{4}}}{2\left(\alpha-1\right)\left(\alpha-2\right)\sqrt{h_n^{\alpha}}}g_{n+2}^{\alpha-2}(x)+R_n^{\alpha}(x),$$

where

$$R_n^{\alpha}(x)=\frac{\left(\frac{\alpha}{2}-\frac{1}{4}\right)\left(\frac{\alpha}{2}-\frac{5}{4}\right)}{\left(\alpha-1\right)\left(\alpha-2\right)\sqrt{h_n^{\alpha}}}\int_{-1}^{x}t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}}g_{n+2}^{\alpha-2}(t)dt. \tag{2.1}$$

Theorem 2.1 (Third aim result)**.** *Suppose that* $\alpha>\frac{9}{2}$*. Then the following asymptotic formula holds*

$$\varphi_{1,1+n}^{\alpha}(x)=\frac{(1-x^2)^{\frac{\alpha}{2}-\frac{1}{4}}}{2\left(\alpha-1\right)\sqrt{h_n^{\alpha}}}g_{n+1}^{\alpha-1}(x)+\frac{(1-x^2)^{\frac{\alpha}{2}-\frac{9}{4}}}{2\left(\alpha-1\right)\left(\alpha-2\right)\sqrt{h_n^{\alpha}}}g_{n+2}^{\alpha-2}(x)+R_n^{\alpha}(x),$$

where $R_n^{\alpha}(x)$ *is given by* (2.1) *and satisfies the estimate*

$$R_n^{\alpha}(x)=o\left(\frac{1}{n}\right).$$

P r o o f. To estimate the remainder $R_n^{\alpha}(x)$, we must consider two cases.

First case: $-1\leq x\leq -1+\frac{1}{n}$. Here we have:

$$g_{n+2}^{\alpha-2}(t)\underset{\vartheta(-1)}{\sim}(-1)^n\frac{\Gamma(2\alpha+n-2)}{\Gamma\left(2\alpha-4\right)\Gamma\left(n+3\right)}\left(1+o\left(x+1\right)\right),$$

then

$$\begin{aligned}|R_n^{\alpha}(x)| &\leq \left|\frac{\left(\frac{\alpha}{2}-\frac{1}{4}\right)\left(\frac{\alpha}{2}-\frac{5}{4}\right)}{\left(\alpha-1\right)\left(\alpha-2\right)\sqrt{h_n^{\alpha}}}\right|\left|\frac{\Gamma(2\alpha+n-2)}{\Gamma\left(2\alpha-4\right)\Gamma\left(n+3\right)}\right|\int_{-1}^{-1+\frac{1}{n}}\left|t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}}\right|dt \\ &\leq \left|\frac{\left(\frac{\alpha}{2}-\frac{1}{4}\right)\left(\frac{\alpha}{2}-\frac{5}{4}\right)}{\left(\alpha-1\right)\left(\alpha-2\right)\sqrt{h_n^{\alpha}}}\right|\left|\frac{\Gamma(2\alpha+n-2)}{\Gamma\left(2\alpha-4\right)\Gamma\left(n+3\right)}\right|\left|1+o\Big(\frac{1}{n}\Big)\right|\int_{-1}^{-1+\frac{1}{n}}\left|t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}}\right|dt \\ &\leq \left|\frac{\left(\frac{\alpha}{2}-\frac{1}{4}\right)\left(\frac{\alpha}{2}-\frac{5}{4}\right)}{\left(\alpha-1\right)\left(\alpha-2\right)\sqrt{h_n^{\alpha}}}\right|\left|\frac{\Gamma(2\alpha+n-2)}{\Gamma\left(2\alpha-4\right)\Gamma\left(n+3\right)}\right|\left|1+o\Big(\frac{1}{n}\Big)\right|\int_{-1}^{-1+\frac{1}{n}}|t|\left|1+t^2\right|^{\frac{\alpha}{2}-\frac{9}{4}}dt.\end{aligned}$$

We let

$$h_n^\alpha = \pi \frac{2^{1-2\alpha}\Gamma(1+2\alpha)}{n!(n+\alpha)\left[\Gamma(\alpha)\right]^2}.$$

It is easy to see that

$$\int_{-1}^{-1+\frac{1}{n}} |t| \left|1+t^2\right|^{\frac{\alpha}{2}-\frac{9}{4}} dt \le \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n^2}\right)^{\frac{\alpha}{2}-\frac{9}{4}} \frac{1}{n} = o\Big(\frac{1}{n}\Big).$$

Then

$$|R_n^\alpha(x)| \le o\Big(\frac{1}{n}\Big) \left| \frac{\left(\frac{\alpha}{2}-\frac{1}{4}\right)\left(\frac{\alpha}{2}-\frac{5}{4}\right)}{(\alpha-1)(\alpha-2)\sqrt{h_n^\alpha}} \right| \left| \frac{\Gamma(2\alpha+n-2)}{\Gamma(2\alpha-4)\Gamma(n+3)} \right| \left| 1+o\Big(\frac{1}{n}\Big) \right| = o\Big(\frac{1}{n}\Big).$$

Second case: $-1+\frac{1}{n} \le x \le 1$. We have:

$$\begin{aligned} |R_n^\alpha(x)| &\le \left| \frac{\left(\frac{\alpha}{2}-\frac{1}{4}\right)\left(\frac{\alpha}{2}-\frac{5}{4}\right)}{(\alpha-1)(\alpha-2)\sqrt{h_n^\alpha}} \right| \int_{-1}^{x} \left| t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}} g_{n+2}^{\alpha-2}(t) \right| dt \\ &\le \omega_{\alpha,n} \left[\int_{-1}^{-1+\frac{1}{n}} \left| t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}} g_{n+2}^{\alpha-2}(t) \right| dt + \int_{-1+\frac{1}{n}}^{x} \left| t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}} g_{n+2}^{\alpha-2}(t) \right| dt \right], \end{aligned}$$

with

$$\omega_{\alpha,n} = \left| \frac{\left(\frac{\alpha}{2}-\frac{1}{4}\right)\left(\frac{\alpha}{2}-\frac{5}{4}\right)}{(\alpha-1)(\alpha-2)\sqrt{h_n^\alpha}} \right|.$$

So

$$\begin{aligned} |R_n^\alpha(x)| &\le \omega_{\alpha,n} \int_{-1}^{-1+\frac{1}{n}} \left| t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}} g_{n+2}^{\alpha-2}(t) \right| dt + \omega_{\alpha,n} \int_{-1+\frac{1}{n}}^{x} \left| t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}} g_{n+2}^{\alpha-2}(t) \right| dt \\ &\le o\Big(\frac{1}{n}\Big) + \omega_{\alpha,n} \int_{-1+\frac{1}{n}}^{x} \left| t\left(1-t^2\right)^{\frac{\alpha}{2}-\frac{9}{4}} g_{n+2}^{\alpha-2}(t) \right| dt. \end{aligned}$$

We say that for each $t=\cos\theta$, $0<\delta\le\theta\le\pi-\delta$, the asymptotic representation is [9, p. 318]

$$P_n^{\left(\alpha-\frac{1}{2},\alpha-\frac{1}{2}\right)}(\cos\theta) = \frac{\cos\left\{(n+\alpha)\theta-\frac{\pi}{2}\right\}}{\sqrt{\pi n}\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\alpha} + o\left(\frac{1}{n^{\frac{3}{2}}}\right),$$

where $P_n^{(\alpha,\beta)}(t)$ is a Jacobi polynomial.

Since

$$g_n^\alpha(t) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma\left(\alpha+n+\frac{1}{2}\right)} P_n^{\left(\alpha-\frac{1}{2},\alpha-\frac{1}{2}\right)}(t),$$

then

$$g_n^\alpha(\cos\theta) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma\left(\alpha+n+\frac{1}{2}\right)} \frac{\cos\left\{(n+\alpha)\theta-\frac{\pi}{2}\right\}}{\sqrt{\pi n}\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\alpha} + o\left(\frac{1}{n^2}\right).$$

First, find an asymptotic estimate for

$$\frac{\Gamma(2\alpha+n)}{\Gamma\left(\alpha+n+\frac{1}{2}\right)}.$$

We see that

$$\Gamma\left(az+b\right)\sim\sqrt{2\pi}e^{-az}\left(az\right)^{az+b-\frac{1}{2}},\quad |\arg z|<\pi,\quad a>0.$$

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim 1+\frac{(b-a)(a+b-1)}{2z}+\frac{1}{12z^2}C_2^{a-b}\left[3(a+b-1)^2-a+b-1\right]+\dots$$

So

$$\begin{aligned}&\frac{\Gamma\left(2\alpha+n\right)}{\Gamma\left(\alpha+n+\frac{1}{2}\right)}\\ &\sim\frac{1}{n^{\alpha-\frac{5}{2}}}+\frac{1}{2n^{\alpha-\frac{3}{2}}}\left(\alpha-\frac{5}{2}\right)\left(3\alpha-\frac{5}{2}\right)+\frac{1}{12n^{\alpha-\frac{1}{2}}}C_2^{\alpha-\frac{5}{2}}\left(3\left(3\alpha-\frac{5}{2}\right)^2-\alpha+\frac{3}{2}\right)+\dots\\ &=o\left(\frac{1}{n}\right),\quad\text{since }\alpha>\frac{9}{2}.\end{aligned}$$

Taking into account the fact that $|1-t^2|\le 1$ for $-1+\frac{1}{n}\le t\le x\le 1$, it follows that:

$$|R_n^\alpha(x)|\le o\left(\frac{1}{n}\right)+\omega_{\alpha,n}\int_{-1+\frac{1}{n}}^{x}|t|\left|g_{n+2}^{\alpha-2}(t)\right|dt,$$

$$\begin{aligned}|R_n^\alpha(x)|\le{}& o\left(\frac{1}{n}\right)\\ &+\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\omega_{\alpha,n}}{\Gamma(2\alpha)\sqrt{\pi n}\left|\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\alpha\right|}\int_{ar\cos(-1+\frac{1}{n})}^{ar\cos(x)}\left|\cos\left\{(n+\alpha)\,\theta-\frac{\pi}{2}\right\}\right|\sin\theta d\theta+o\left(\frac{1}{n^2}\right).\end{aligned}$$

Now, using the properties of the trigonometric functions, we get

$$\begin{aligned}|R_n^\alpha(x)|&\le o\left(\frac{1}{n}\right)+o\left(\frac{1}{n}\right)\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\omega_{\alpha,n}}{\Gamma(2\alpha)\sqrt{\pi n}\left|\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\alpha\right|}\int_{ar\cos(-1+\frac{1}{n})}^{ar\cos(x)}\theta d\theta+o\left(\frac{1}{n^2}\right)\\ &\le o\left(\frac{1}{n}\right)+o\left(\frac{1}{n}\right)\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\omega_{\alpha,n}}{\Gamma(2\alpha)\sqrt{\pi n}\left|\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\alpha\right|}\int_{ar\cos(-1+\frac{1}{n})}^{\pi}\theta d\theta+o\left(\frac{1}{n^2}\right)\\ &\le o\left(\frac{1}{n}\right)+o\left(\frac{1}{n}\right)\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\omega_{\alpha,n}}{\Gamma(2\alpha)\sqrt{\pi n}\left|\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\alpha\right|}\left(\frac{\pi^2}{2}-\frac{ar\cos^2(-1+\frac{1}{n})}{2}\right)+o\left(\frac{1}{n^2}\right)\\ &=o\left(\frac{1}{n}\right).\end{aligned}$$

So, we have the desired estimate. □

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